Runge - Kutta Method

The method is named after two German mathematicians Carl Runge and Wilhelm Kutta. It was developed to avoid the computation of higher order derivations which the Taylor's method may involve.

The Runge-kutta formulas for several types are given below:

1- Fourth Order Runge - Kutta Method

Let $\frac{dy}{dx} = f(x, y)$

represents any first order differential equation and let h denotes the step length. If x_0, y_0 denote the initial values, then the the value of y_1 corresponding to the value of $x_1 = x_0 + h$ is computed from the formula:

$$y_1 = y_0 + \Delta y$$

Where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $k_1 = h \cdot f(x_0, y_0)$.

$$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

$$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$$

is most commonly used and is known as Runge-Kutta method only.

Example: Use Runge-Kutta method to solve the equation:

$$\frac{dy}{dx} = x + y \quad \text{at} \quad x = 0.1$$

given the initial condition:
$y = 1$ when $x = 0$

solution:

we have: $f(x,y) = x + y$
$x_0 = 0$
$y_0 = 1$
$h = 0.1$
$x_1 = 0.1$
$y_1 = ?$

$x = x_0 + h$
$0.1 = 0 + h$
$h = 0.1$

$$k_1 = h \cdot f(x_0, y_0)$$

$$= 0.1 * (x_0 + y_0) = 0.1 * (0 + 1) = 0.1$$

$$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h \cdot f(0 + 0.05, 1 + 0.05)$$

$$= 0.1 \cdot f(0.05, 1.05)$$
$$= 0.1 * (0.05 + 1.05)$$
$$= 0.11$$

$$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 * f(0 + 0.05, 1 + 0.055)$$

$$= 0.1 * f(0.05, 1.055)$$

$$= 0.1 * (0.05 + 1.055)$$

$$= 0.1 * 1.105$$

$$= 0.1105$$

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$$

$$= 0.1 * f(0 + 0.1, 1 + 0.1105)$$

$$= 0.1 * f(0.1, 1.1105)$$

$$= 0.1 * (0.1 + 1.1105)$$

$$= 0.1 * 1.2105$$

$$= 0.12105$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.22 + 0.221 + 0.12105)$$

$$= 0.11034$$

$$\therefore y_1 = 1 + 0.11034$$

$$= 1.11034$$

Example :

Using Rung-Kuttu Method solve the equation :

$$\frac{dy}{dx} = x + y^2 \quad \text{at} \quad x = 0.2$$

given $y = 1$ when $x = 0$ taking step-length $h = 0.1$.

Solution :

we have: $f = x + y^2$

$x_0 = 0$

$y_0 = 1$

$h = 0.1$

$x_1 = 0.1$

$y_1 = ?$

$x_2 = 0.2$

$y_2 = ?$

$$K_1 = h \cdot f(x_0, y_0)$$
$$= 0.1 * (0 + 1) = 0.1$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) \qquad x + y^2$$

$$= 0.1 * (0.05 + 1.1025) = 0.11525$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= 0.1 * (0.05 + 1.1185) = 0.11685$$

$$K_4 = h \cdot f(x_0 + h, y_0 + K_3)$$
$$= 0.1 * (0.1 + 1.2474) = 0.13474$$

$\therefore \quad \Delta y = \frac{1}{6} \left(0.1 + 2(0.11525) + 2(0.11685) + 0.13474 \right)$

$= 0.1165$

$\therefore \quad y_1 = y_0 + \Delta y = 1 + 0.1165$

$= 1.1165$

For the second step we have:

$x_0 = 0.1 \qquad y_0 = 1.1165$

$k_1 = 0.1(0.1 + 1.2466) = 0.1347$

$k_2 = 0.1(0.15 + 1.4014) = 0.1551$

$k_3 = 0.1(0.15 + 1.4259) = 0.1576$

$k_4 = 0.1(0.2 + 1.6233) = 0.1823$

$\therefore \quad \Delta y = \frac{1}{6}(0.9424) = 0.1571$

$\therefore \quad y(0.2) = 1.1165 + 0.1571$

$= 1.2736$

Home Work:

1- Draw a flow chart for the Runge-Kutta Method.

2. Use Runge-kutta method to solve the equation:

$$\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$$ at $x = 0.1, 0.2, 0.3$ and 0.4

given that $y = 1$ when $x = 0$

Answer : 1.0874 , 1.1557 , 1.2104 , 1.2544

3. Use Runge - Kutta method to solve the equation :

$$\frac{dy}{dx} = -xy$$ at $x = 0.2$

given that $y = 1$ when $x = 0$ and $h = 0.2$

Answer : 0.9802

4. Use Runge - Kutta method to solve the equation :

$$\frac{dy}{dx} = \sqrt{x + y}$$ at $x = 0.8$

given that $y = 0.41$ when $x = 0.4$ using $h = 0.4$

Answer : 1.1678